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Abstract

An asymptotic theory is developed for general non-integrable boundary quantum field theory in 1 + 1 dimensions based on the Lagrangian description. Reflection matrices are defined to connect asymptotic states and are shown to be related to the Green functions via the boundary reduction formula derived. The definition of the R-matrix for integrable theories due to Ghoshal and Zamolodchikov and that used in the perturbative approaches are shown to be related.

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1. Introduction

Two-dimensional boundary quantum field theories have been analysed from two different points of view, the bootstrap and the perturbative, respectively.

The former was initiated by Ghoshal and Zamolodchikov in [1] and can be applied to integrable theories. In such theories there is an infinite number of conserved quantities, which give severe restrictions on the allowed physical processes. Besides the usual constraints such as factorization and purely elastic bulk scattering there is also factorization and purely elastic reflection on the boundary. The scattering theory developed in [1] is analogous to the axiomatic scattering theory [15]: in the *in* state the particles travel towards the boundary with decreasingly ordered momenta, while in the *out* state, where all the scatterings and reflections have been terminated, they travel away from the boundary with decreasingly ordered momenta again. The R-matrix which connects the *in* and *out* states is the composition of the individual reflection and the pairwise scattering matrices. The one-particle reflection matrices have to obey unitarity, boundary Yang–Baxter and boundary crossing relations. Using these relations together with the bootstrap condition [1, 2] the model can be solved modulo CDD type ambiguities. We emphasize that in the bulk case the axioms of the scattering theory such as unitarity, crossing symmetry [15] were motivated by relativistic field theoretic results based on the perturbative, Lagrangian description. In the boundary case, however, to our knowledge, no such background is available. In [3] the nonlinear Schrödinger model with linear boundary condition on the half line is considered and the assumptions of the axiomatic scattering theory are rigorously checked. This model is, however, non-relativistic and integrable.

The perturbative approach to boundary quantum field theories can be applied without the assumption of integrability. It was started with the analysis of bulk perturbation [4, 5] with the Neumann boundary condition, then extended to boundary perturbations in [6–12]. Most of these papers deal with comparing exact results, obtained in the aforementioned way for the reflection matrices in integrable theories, on one hand, and perturbative results on the other. They defined the reflection matrix $R(k)$ through the asymptotic behaviour of the two-point function of the field Φ—creating the particles—far away from the boundary:

$$\langle 0|T(\Phi(x,t)\Phi(x',t'))|0\rangle = \int \frac{\mathrm{d}\omega}{2\pi}\frac{\mathrm{e}^{-\mathrm{i}\omega(t-t')}}{2k(\omega)}(\mathrm{e}^{\mathrm{i}k(\omega)|x-x'|} + R(k)\,\mathrm{e}^{-\mathrm{i}k(\omega)(x+x')}) + \cdots. \tag{1}$$

No detailed justification has been given for this quantity being the same as the axiomatic R-matrix.

In this paper, we develop the asymptotic theory for a scalar field with the most general bulk and boundary self-interaction. We derive the boundary reduction formula, which connects the reflection matrix to the two-point function. As a consequence we can fill the gap mentioned above, that is we are able to connect the R-matrix of the axiomatic theory to (1).

The paper is organized as follows: we apply the canonical quantization procedure to the free theory, in which case the boundary condition is Neumann. The interacting theory is defined by means of the adiabatic hypothesis. Asymptotic states and reflection matrices are introduced and the simplest physical process of one incoming particle is demonstrated. As the main result we derive the boundary reduction formula. Having developed the boundary perturbation theory we are able to connect the earlier definitions of the R-matrix, and finally we conclude on their equivalence. A brief explanation of the Feynman rules is given in appendix A, while the structure of the two-point function is analysed in appendix B.

2. The free theory

The system we are dealing with contains a real scalar field $\Phi(x,t)$, living on the half space $x \leqslant 0$. The bulk and boundary interactions are described by the action

$$S = \int_{-\infty}^{\infty}\mathrm{d}t\int_{-\infty}^{0}\mathrm{d}x\left[\frac{1}{2}((\partial_t\Phi)^2 - (\partial_x\Phi)^2 - m^2\Phi^2) - V(\Phi)\right] - \int_{-\infty}^{\infty}\mathrm{d}t\,U(\Phi(0,t)). \tag{2}$$

The free theory can be obtained by switching off the bulk and the boundary interactions: $V(\Phi) = U(\Phi) = 0$. The equation of motion is the usual bulk free equation, the boundary condition is, however, the Neumann one

$$(\Box + m^2)\Phi(x,t) = 0 \qquad \partial_x\Phi(x,t)|_{x=0} = 0.$$

In solving these equations by Fourier transformation we have to use the complete system of functions with this boundary condition

$$\Phi(x,t) = \int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}\cos(kx)\tilde{\Phi}(k,t) \qquad \tilde{\Phi}(k,t) = \tilde{\Phi}(-k,t).$$

The conjugate momentum also satisfies Neumann boundary condition so the canonical commutation relation reads

$$[\Phi(x,t),\Pi(x',t)] = \mathrm{i}\delta_N(x,x') \equiv \mathrm{i}\delta(x-x') + \mathrm{i}\delta(x+x').$$

The creation and annihilation operators which diagonalize the Hamiltonian are

$$a(k,t) = \mathrm{i}\tilde{\Pi}(k,t) + \omega(k)\tilde{\Phi}(k,t) \qquad a^{+}(k,t) = -\mathrm{i}\tilde{\Pi}(k,t) + \omega(k)\tilde{\Phi}(k,t)$$

where $\omega(k) = \sqrt{k^2 + m^2}$. Their commutation relations are

$$[a(k,t), a^+(k',t)] = 2\pi 2\omega(k)(\delta(k-k') + \delta(k+k')).$$

Normal ordering is defined as usual: creation operators $a^+(k,t)$ are to the left of annihilation operators $a(k',t)$. Since the normal ordered Hamiltonian is

$$H = \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}\tilde{k}\, \omega(k) a^+(k,t) a(k,t) \qquad \mathrm{d}\tilde{k} = \frac{\mathrm{d}k}{2\pi 2\omega(k)}$$

the time dependence can be determined exactly: $a^+(k,t) = a^+(k)\,\mathrm{e}^{\mathrm{i}\omega(k)t}$ and $a(k,t) = a(k)\,\mathrm{e}^{-\mathrm{i}\omega(k)t}$. Putting it back into the expansion of Φ gives rise to

$$\Phi(x,t) = \int_{-\infty}^{\infty} \mathrm{d}\tilde{k}\, \cos(kx)(a^+(k)\,\mathrm{e}^{\mathrm{i}\omega(k)t} + a(k)\,\mathrm{e}^{-\mathrm{i}\omega(k)t}). \tag{3}$$

The Fock Hilbert space $\mathcal{H}$ can be built up by the action of the creation operators on the vacuum:

$$a(k)|0\rangle = 0 \qquad \forall k$$
$$|k_1, k_2, \ldots, k_n\rangle = a^+(k_1)a^+(k_2)\cdots a^+(k_n)|0\rangle \qquad k_1 \geqslant k_2 \geqslant \cdots \geqslant k_n \geqslant 0.$$

Note that in labelling the states, k is always positive[3]. For technical reasons in some formulae we also allow k to take negative values, but we always mean a symmetric extension, that is $a(k) = a(-k)$. The vacuum expectation value of the time-ordered product

$$\langle 0|T(\Phi(x,t)\Phi(x',t'))|0\rangle = \int_{-\infty}^{\infty} \frac{\mathrm{d}^2k}{(2\pi)^2} \frac{\mathrm{i}\,\mathrm{e}^{-\mathrm{i}k_0(t-t')}}{k^2 - m^2 + \mathrm{i}\epsilon} (\mathrm{e}^{\mathrm{i}k_1(x-x')} + \mathrm{e}^{\mathrm{i}k_1(x+x')}) \tag{4}$$

solves the inhomogeneous equation

$$(\Box + m^2)\langle 0|T(\Phi(x,t)\Phi(x',t'))|0\rangle = -\mathrm{i}\delta_N(x,x')\delta(t-t').$$

Besides the usual bulk propagator, which describes how the field propagates from (x,t) to (x',t'), (4) also contains another contribution, which can be interpreted as a bulk propagation of the field from $(-x,t)$ to (x',t'). Thus the free boundary theory (Neumann boundary condition) can be realized by the mirror trick: we compute every quantity in the usual bulk theory, but any time we insert a field at (x,t) we insert the same type of field also at the mirror point $(-x,t)$. Since the interacting theory is defined in terms of the free quantities, (in the calculations we use the free propagator) we have the following interpretational consequence: the particles interact not only with themselves but also with their mirror partners.

3. Interacting theory, asymptotic states

Non-trivial interaction is described by (2) when $U(\Phi)$, $V(\Phi)$ or both are nonzero. To handle this case we use the adiabatic hypothesis. That is the interaction is switched on adiabatically in the remote past and switched off in the remote future. Moreover, we also suppose that the particle spectrum does not change during this adiabatic procedure: only the masses are renormalized. In a real scattering experiment the prepared state is one of the free theory (*in* state) and the detected state is also a free state (*out* state). Both the *in* and the *out* states provide a basis for the Hilbert space $\mathcal{H}$, and the R-matrix is a unitary transformation connecting the two:

$$|\text{final}\rangle_{out} = R|\text{initial}\rangle_{in}.$$

The unitarity of the R-matrix expresses the fact that the transition probabilities sum up to one.

[3] Actually k provides only a different parametrization of the energy by the relation $k = \sqrt{\omega^2 - m^2}$, since in the presence of a boundary the momentum is not conserved.

The simplest process we can imagine is that in which a single particle travels toward the boundary, reflects on it and then returns as a multi-particle configuration. The naive bulk analogue of this process is trivial since asymptotic particles are stable by assumption. In the presence of the boundary we can interpret this process in terms of the mirror transformation. In this language the *in* state contains not only the incoming particle but also its mirror image with respect to the boundary $x = 0$. The scattering of the incoming particle on the boundary has a contribution describing its scattering on its mirror image, a process analogous to the two-particle scattering in the bulk. To be concrete: in the initial state of this process we have a wave packet of the form

$$|\text{initial}\rangle_{in} = \int_{-\infty}^{\infty} \mathrm{d}\tilde{k}' f(k')|k'\rangle_{in} \tag{5}$$

where we suppose that $f(k')$ is well localized around k. The spacetime dependence of such a configuration is

$$\tilde{f}(x,t) = \int_{-\infty}^{\infty} \tilde{\mathrm{d}}k' f(k') \cos(k'x)\,\mathrm{e}^{-\mathrm{i}\omega(k')t}.$$

It describes a wave packet travelling with momentum k towards the boundary. It also contains, however, the mirror image of the packet which is on the other side of the wall (so is not in the real spacetime) and travels with momentum $-k$ as shown in the following figure:

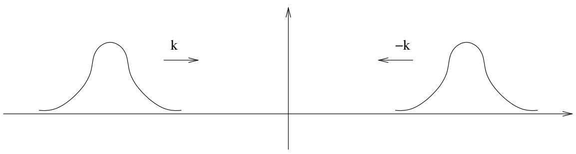

Figure 1. Initial state.

If there is no interaction (free case) then this state (5) is the eigenstate of the free Hamiltonian. Since the time evolution is trivial the picture in the remote future looks like

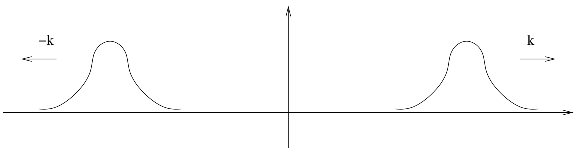

Figure 2. Final state.

Now the real and reflected particle travels away from the boundary with momentum $|-k|$ and the mirror image with momentum k.

In the interacting case the final state may contain particles (or just one particle in the integrable case) travelling backward from the boundary. This coincides with the idea of [1] where the *in* state contains a particle with rapidity θ while the *out* state has rapidity $-\theta$.

4. Boundary reduction formula

We are interested in the case when both the *in* and the *out* states contain a single particle with definite energy. The energy conservation can be factored out:

$${}_{out}\langle k'|k\rangle_{in} = 2\pi(\delta(k-k') + \delta(k+k'))\omega(k)\mathcal{R}(|k|).$$

Our aim is to make correspondence with the other definitions of the reflection matrix. For this reason we express the reflection matrix $\mathcal{R}$ in terms of the correlation functions. In the bulk theory this is done using the reduction formula [14]. In the following, we derive an analogous formula for boundary theories. The steps of the derivation are similar to those in [14].

The *in* field is a free field so we can use the decomposition (3). The inverse of this relation is

$$
\begin{aligned}
a_{in}(k) &= 2\mathrm{i}\int_{-\infty}^{0}\mathrm{d}x\,\cos(kx)\,\mathrm{e}^{\mathrm{i}\omega(k)t}\overset{\leftrightarrow}{\partial}_t\,\Phi_{in}(x,t)\\
a_{in}^{+}(k) &= -2\mathrm{i}\int_{-\infty}^{0}\mathrm{d}x\,\cos(kx)\,\mathrm{e}^{-\mathrm{i}\omega(k)t}\overset{\leftrightarrow}{\partial}_t\,\Phi_{in}(x,t).
\end{aligned}
\tag{6}
$$

Using the definition of the *in* state we have

$$
{}_{out}\langle p_1,\ldots,p_k|q_1,\ldots,q_l\rangle_{in} =: \langle\,\rangle = {}_{out}\langle p_1,\ldots,p_k|a_{in}^{+}(q_1)|q_2,\ldots,q_l\rangle_{in}. \tag{7}
$$

Now apply formulae (6) to obtain

$$
\langle\,\rangle = -2\mathrm{i}\int_{-\infty}^{0}\mathrm{d}x\,\cos(q_1x)\,\mathrm{e}^{-\mathrm{i}\omega(q_1)t}\overset{\leftrightarrow}{\partial}_t\,{}_{out}\langle p_1,\ldots,p_k|\Phi_{in}(x,t)|q_2,\ldots,q_l\rangle_{in}.
$$

We suppose that the *in* field can be expressed in terms of the interacting field as $\Phi(x,t)\to Z^{1/2}\Phi_{in}(x,t)$ as $t\to-\infty$. As a consequence

$$
\langle\,\rangle = -\mathrm{i}\lim_{t\to-\infty}Z^{-1/2}2\int_{-\infty}^{0}\mathrm{d}x\,\cos(q_1x)\,\mathrm{e}^{-\mathrm{i}\omega(q_1)t}\overset{\leftrightarrow}{\partial}_t\,{}_{out}\langle p_1,\ldots,p_k|\Phi(x,t)|q_2,\ldots,q_l\rangle_{in}.
$$

Since $\lim_{t\to\infty}\Phi(x,t) = \lim_{t\to\infty}Z^{1/2}\Phi_{out}(x,t)$ we also have

$$
\begin{aligned}
\langle\,\rangle = {}&{}_{out}\langle p_1,\ldots,p_k|a_{out}^{+}(q_1)|q_2,\ldots,q_l\rangle_{in} + \mathrm{i}Z^{-1/2}2\int_{-\infty}^{0}\mathrm{d}x\int_{-\infty}^{\infty}\mathrm{d}t\,\partial_t\{\cos(q_1x)\\
&\times\mathrm{e}^{-\mathrm{i}\omega(q_1)t}\overset{\leftrightarrow}{\partial}_t\,{}_{out}\langle p_1,\ldots,p_k|\Phi(x,t)|q_2,\ldots,q_l\rangle_{in}\}
\end{aligned}
$$

from which the connected part is

$$
\begin{aligned}
&\mathrm{i}Z^{-1/2}2\int_{-\infty}^{0}\mathrm{d}x\int_{-\infty}^{\infty}\mathrm{d}t\,\mathrm{e}^{-\mathrm{i}\omega(q_1)t}\left\{\cos(q_1x)\langle out|\partial_t^2\Phi(x,t)|in\rangle\right.\\
&\qquad\left.+\langle out|\Phi(x,t)|in\rangle\left(-\partial_x^2+m^2\right)\cos(q_1x)\right\}
\end{aligned}
$$

where $\langle out|$ $(|in\rangle)$ is the shorthand form for $\langle p_1,\ldots,p_k|$ $(|q_2,\ldots,q_l\rangle)$, respectively. Performing the partial integration, (which is legitimate if momenta are smeared with wave packets of the form of (5)), we have to be careful to keep the surface term. The connected part turns out to be

$$
\mathrm{i}Z^{-1/2}2\int\mathrm{d}^2x\,\mathrm{e}^{-\mathrm{i}\omega(q_1)t}\cos(q_1x)\{\Box+m^2+\delta(x)\partial_x\}\langle out|\Phi(x,t)|in\rangle
$$

where $\int\mathrm{d}^2x = \int_{-\infty}^{0}\mathrm{d}x\int_{-\infty}^{\infty}\mathrm{d}t$ is the integral over the entire physical spacetime. This is the first stage of the reduction formula. In the second step we eliminate an outgoing particle. The derivation straightforwardly follows the combination of the previous computation and the usual bulk derivation. The connected part of the result is

$$
\begin{aligned}
{}_{out}\langle p_1,\ldots,p_k|q_1,\ldots,q_l\rangle_{in} &= -4Z^{-1}\int\mathrm{d}^2x\,\mathrm{d}^2x'\,\mathrm{e}^{\mathrm{i}(\omega(p_1)t'-\omega(q_1)t)}\cos(q_1x)\cos(p_1x')\\
&\{\Box+m^2+\delta(x)\partial_x\}\{\Box'+m^2+\delta(x')\partial_{x'}\}\langle p_2,\ldots,p_n|T(\Phi(x,t)\Phi(x',t'))|in\rangle.
\end{aligned}
$$

Iterating the above steps the general matrix element (7) can be expressed in terms of the $(k+l)$-point function.

In particular, for the reflection matrix we have

$$
\begin{aligned}
{}_{out}\langle k'|k\rangle_{in} - {}_{out}\langle k'|k\rangle_{out} &= 2\pi(\delta(k-k')+\delta(k+k'))\omega(k)(\mathcal{R}(|k|)-1) \\
&= -4Z^{-1}\int \mathrm{d}^2x\,\mathrm{d}^2x'\,\mathrm{e}^{\mathrm{i}(\omega(p_1)t'-\omega(q_1)t)}\cos(q_1x)\cos(p_1x') \\
&\quad\times\{\Box+m^2+\delta(x)\partial_x\}\{\Box'+m^2+\delta(x')\partial_{x'}\}G(x,x',t-t') \qquad (8)
\end{aligned}
$$

where

$$
G(x,x',t-t') = \langle 0|T(\Phi(x,t)\Phi(x',t'))|0\rangle. \qquad (9)
$$

5. Perturbation theory

Let us turn to the description of the interacting theory as a perturbation of the free one. In doing so we use the interaction representation. That is the time evolution operator is given by the time-ordered (T) product as

$$
U(t) = T\exp\left\{-\mathrm{i}\int_{-\infty}^{t}\mathrm{d}t'\,H_{\mathrm{int}}(t')\right\}.
$$

This Hamiltonian contains the *in* fields and acts on the *in* Hilbert space by construction. Clearly the R-matrix can be expressed as

$$
R = U(\infty) = T\exp\left\{-\mathrm{i}\int_{-\infty}^{\infty}\mathrm{d}t'\,H_{\mathrm{int}}(t')\right\}
$$

which also gives a direct calculation of this quantity. The interacting field is built up from the free field as

$$
\Phi(x,t) = U^{-1}(t)\Phi_{in}(x,t)U(t).
$$

Putting this expression into the two-point function (9) and using the usual heuristic derivation we obtain

$$
\langle 0|T(\Phi(x,t)\Phi(x',t'))|0\rangle = \frac{\langle 0|T(\Phi_{in}(x,t)\Phi_{in}(x',t')\exp\{\mathrm{i}\int\mathrm{d}^2x\,\mathcal{L}_{\mathrm{int}}[\Phi_{in}(x,t)]\})|0\rangle}{\langle 0|T(\exp\{\mathrm{i}\int\mathrm{d}^2x\,\mathcal{L}_{\mathrm{int}}[\Phi_{in}(x,t)]\})|0\rangle}.
$$

Expanding the exponential we arrive at the perturbative series

$$
= \frac{\sum_{n=0}^{\infty}\frac{\mathrm{i}^n}{n!}\langle 0|T(\Phi_{in}(x,t)\Phi_{in}(x',t')\int\mathrm{d}^2x_1\,\mathcal{L}_{\mathrm{int}}[\Phi_{in}(x_1,t_1)]\cdots\int\mathrm{d}^2x_n\,\mathcal{L}_{\mathrm{int}}[\Phi_{in}(x_n,t_n)])|0\rangle}{\sum_{n=0}^{\infty}\frac{\mathrm{i}^n}{n!}\langle 0|T(\int\mathrm{d}^2x_1\,\mathcal{L}_{\mathrm{int}}[\Phi_{in}(x_1,t_1)]\cdots\int\mathrm{d}^2x_n\,\mathcal{L}_{\mathrm{int}}[\Phi_{in}(x_n,t_n)])|0\rangle}. \qquad (10)
$$

In computing the vacuum expectation values of the product of the fields we can use Wick's theorem. The results are encoded in the Feynman rules which are given in appendix A.

From careful analysis of the perturbative series (see appendix B) one can deduce that the momentum space Green function has the following form:

$$
G(p,p',\omega) = 2\pi(\delta(p+p')+\delta(p-p'))G_{\mathrm{bulk}}(p,\omega) + G_{\mathrm{bulk}}(p,\omega)B(p,p',\omega)G_{\mathrm{bulk}}(p',\omega). \qquad (11)
$$

Here $G_{\mathrm{bulk}}(p,\omega)$ is the propagator of the bulk theory, which in terms of the spectral function $\sigma(m^2)$ has the usual Källen–Lehmann representation [14]

$$
G_{\mathrm{bulk}}(p,\omega) = \frac{\mathrm{i}Z}{\omega^2-p^2-m^2+\mathrm{i}\epsilon} + \int_{4m^2}^{\infty}\mathrm{d}m'^2\frac{\mathrm{i}\sigma(m'^2)}{\omega^2-p^2-m'^2+\mathrm{i}\epsilon}. \qquad (12)
$$

We also have the decomposition

$$B(p,p',\omega)=B_1(p,p',\omega)+B_2(p,\omega)+B_2(p',\omega)+B_3(\omega). \tag{13}$$

The interpretation of the terms in (11) is the following: the first term describes the propagation in the presence of the boundary without hitting the boundary. In the second term $G_{\mathrm{bulk}}(p,\omega)$ is the propagator to the boundary, $B(p,p',\omega)$ is the reflection on the boundary, while $G_{\mathrm{bulk}}(p',\omega)$ describes the propagation back from the boundary. In the reflection matrix $B_1(p,p',\omega)$ really depends on both momenta and comes from the purely bulk interactions, $B_3(\omega)$ is the purely boundary contribution and B_2 represents the cross terms.

Now we are able to relate the two different definitions of the R-matrix. Performing both inverse Fourier transformations of (11) in the momentum variables, but keeping only the contributions of the poles of the first term in propagators (12)

$$G(x,x',t-t')=\int\frac{\mathrm{d}\omega}{2\pi}\frac{Z\,\mathrm{e}^{-\mathrm{i}\omega(t-t')}}{2k(\omega)}\left(\mathrm{e}^{\mathrm{i}k(\omega)|x-x'|}+\left(1+\frac{ZB(k(\omega),k(\omega),\omega)}{2k(\omega)}\right)\mathrm{e}^{-\mathrm{i}k(\omega)(x+x')}\right)$$

where $k(\omega)=\sqrt{\omega^2-m^2}$. Comparing the reflected wave with the unreflected one the reflection matrix was defined to be

$$R(k)=1+\frac{ZB(k(\omega),k(\omega),\omega)}{2k(\omega)}.$$

We will recover the same result from our boundary reduction formula (8). First we recall that the reduction formula describes the way the matrix elements are related to the correlation functions. Considering the correlation functions in momentum space the operator $\Box+m^2$ gives a factor of $-k^2+m^2$ for each external leg. The spacetime integrations, as inverse Fourier transformations, put all the momenta on shell. Since all of the outer legs in the correlation functions are dressed up in the perturbation theory to contain the exact bulk propagators (12) with poles of the form $\frac{\mathrm{i}Z}{k^2-m^2}$, the reduction formula merely amputates the legs and gives the residue of this multi-pole. In the boundary case we have an analogous interpretation. Similar to the bulk case the momentum conserving part of (11) does not give any contribution to the R-matrix so it is enough to consider the other term. A careful analysis shows that $(\Box+m^2)$ is the operator which amputates the legs starting with a bulk vertex, while $\delta(x)\partial_x$ is responsible for amputation of the legs starting with a boundary vertex. As a consequence the $(\Box+m^2)(\Box'+m^2)$ term gives $B_1(k,k',\omega(k))$, the terms $(\Box+m^2)\delta(x')\partial_{x'}$ and $(\Box'+m^2)\delta(x)\partial_x$ together give $B_2(k,\omega(k))+B_2(k',\omega(k))$, finally $\delta(x)\partial_x\delta(x')\partial_{x'}$ gives $B_3(\omega(k))$. We also have an overall factor $2\pi Z\delta(\omega(k)-\omega(k'))$ expressing energy conservation. Collecting all these terms and using the identity $2\pi\delta(\omega(k)-\omega(k'))=\frac{\omega(k)}{k}2\pi(\delta(k-k')+\delta(k+k'))$, we obtain that

$$\mathcal{R}(k)=1+\frac{ZB(k,k,\omega(k))}{2k}$$

which shows that the reflection factor defined by using asymptotic states and that defined using the two-point function are identical.

6. Conclusion

The boundary reduction formula, derived in the paper for a boson with the most general (possibly non-integrable) boundary and bulk self-interaction, showed the equivalence of the previously used definitions for the R-matrix, and supports the correctness of the formalism. This formulation enables one to derive the main properties of the R-matrix such as analyticity, unitarity, crossing symmetry and analyse its pole structure directly without referring to the crossed channel picture used in [1]. The analysis of the perturbative series, Landau equations, Cutkosky rules, the derivation of the boundary Coleman–Thun mechanism and of the analyticity properties of the R-matrix are the subjects of our next paper.

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Appendix A

The Feynman rules of the boundary theory can be obtained by expanding the interaction Lagrangian as

$$\mathcal{L}_{\mathrm{int}}(x,t) = \sum_{N=0}^{\infty} \frac{\alpha_N}{N!} \Phi(x,t)^N + \delta(x) \sum_{M=0}^{\infty} \frac{\beta_M}{M!} \Phi(0,t)^M \tag{14}$$

and putting (14) into (10). In contrast to the perturbative approaches developed so far we formulate the resulting Feynman rules in momentum space. That is we compute

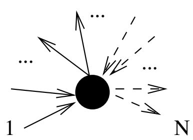
$$\tilde{G}(p_1,\omega_1,\ldots,p_n,\omega_n) = \int_{-\infty}^{\infty} \mathrm{d}x_1 \cdots \int_{-\infty}^{\infty} \mathrm{d}x_n \times \int_{-\infty}^{\infty} \mathrm{d}t_1 \cdots \int_{-\infty}^{\infty} \mathrm{d}t_n \, \mathrm{e}^{\mathrm{i}\sum_{j=1}^{n}(p_j x_j - \omega_j t_j)} G(x_1,t_1,\ldots,x_n,t_n)$$

according to the Feynman rules:

- Draw all possible oriented graphs with n external legs such that the external legs point inside the graph. The lines (both external and internal) can be of two types: straight and dashed. The vertices can be of two types: black and white. Label the external legs by the 2-momenta $(p_1,\omega_1)\ldots(p_n,\omega_n)$, and the internal lines by new momentum variables (k^1,k^0).

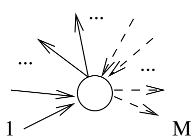
- To a black vertex with N incident lines attach the contribution

$$\mathrm{i}\alpha_N 2\pi\delta\left(\sum_{i\in in} k_i^0 - \sum_{i\in out} k_i^0\right)\pi\delta\left(\sum_{i\in in}^{\mathrm{str}} k_i^1 - \sum_{i\in in}^{\mathrm{dsh}} k_i^1 - \sum_{i\in out} k_i^1\right). \tag{15}$$



- To a white vertex with M incident lines attach the contribution

$$\mathrm{i}\beta_M 2\pi\delta\left(\sum_{i\in in} k_i^0 - \sum_{i\in out} k_i^0\right). \tag{16}$$



- To a line of any type labelled by (k^1, k^0) attach the contribution

$$\frac{\mathrm{i}}{k^2 - m^2 + \mathrm{i}\epsilon}. \tag{17}$$

If the line is internal then integrate over k: $\int \frac{\mathrm{d}^2 k}{(2\pi)^2}$.

- Sum over all topologically distinct diagrams.

The notation $i \in in/out$ means that the line labelled by $\left(k_i^1, k_i^0\right)$ is incoming/outgoing at the given vertex. The label str/dsh on the sum means that the summation goes only over straight/dashed lines.

These Feynman rules need some explanation. The derivation is fairly standard: one derives the graph rules first in coordinate space. This is done by writing $G(x_1, t_1 \ldots x_n, t_n)$ as a perturbative series. Each term of the series is represented by a graph associating the various parts of the contribution with the vertices and lines of the graph. In particular, there is a spacetime integration associated with each vertex. According to the two terms in (14), there are two kinds of vertices: black (bulk) ones corresponding to the first term and white (boundary) ones corresponding to the second term. Owing to the presence of the factor $\delta(x)$ in the second term of (14) at the boundary vertices the space integration can be performed. There is only a single type of graph line at this stage.

Converting to momentum space (Fourier transformation of $G(x_1, t_1 \ldots x_n, t_n)$), all the remaining spacetime integrations can be performed resulting in δ functions at each vertex on the momenta of the lines incident on the vertex. At the boundary vertex there is only a time integration left, hence we obtain a δ function involving only the time-like component of the momenta. At the bulk vertex there are both time and space integrations resulting in two δ functions. The presence of both $(x_i - x_j)$- and $(x_i + x_j)$-dependent terms in the free propagator (4) implies that we have a sum of δ functions on the space-like component of the momenta. The δ in this sum differ in the signs of some momenta in the argument. The two types of lines are introduced in order to associate different graphs with the individual terms: a line is straight if it corresponds to the term coming from the $(x_i - x_j)$-dependent part of the free propagator associated with the given line, and dashed if it comes from the $(x_i + x_j)$-dependent part.

Appendix B

In this section we analyse the properties of the two-point function in momentum space. A systematic investigation of the correlation functions can be achieved by generalizing the parametric representation ([14], section 6.2.3) to the boundary case. Since the introduction of all the machinery is quite lengthy and we need the result only for the two-point function, instead of the detailed presentation we summarize how formula (11) can be obtained by the direct analysis of the perturbative series.

Energy is conserved at each vertex so we have

$$\tilde{G}(p_1, \omega_1, p_2, \omega_2) = 2\pi\delta(\omega_1 - \omega_2)G(p_1, p_2, \omega_1).$$

Momentum is not conserved in general. There are graphs whose contribution spoils momentum conservation: such is any graph containing a boundary vertex. On the other hand, a graph with only bulk vertices and straight lines gives rise to a contribution proportional to $\delta(p_1 - p_2)$. Furthermore, the contribution is proportional to the contribution of the same graph computed in the bulk theory. This can be verified by comparing the appropriate graph rules with the bulk theory rules, and one finds that the only difference is a factor $1/2$ in the bulk vertex contribution. That is the contribution of a graph with N_b bulk vertices, no boundary vertices and only straight lines is 2^{-N_b} times the contribution of the same graph in the bulk theory.

On the other hand, there is a symmetry operation on the graphs that leaves the contribution invariant. Namely, at each bulk vertex we can change the types of all incident lines and also the signs of the momenta labelling the *outgoing* lines. One checks that this transformation—performed independently at each of the N_b bulk vertices—does not affect the graph contribution and gives rise to a symmetry factor 2^{N_b}. This symmetry factor just compensates the factor 2^{-N_b}. Hence the sum of the contributions of the graphs with only bulk vertices and straight lines plus those related to them by the symmetry transformation described gives the propagator of the bulk theory. It is not difficult to see that no other graph gives a contribution respecting momentum conservation.

Summing then the contributions of the graphs with only bulk vertices and straight *internal* lines (and arbitrary *external* ones) plus the graphs related to them by symmetry transformation we obtain the 'momentum-preserving' part of the propagator

$$2\pi(\delta(p_1 + p_2) + \delta(p_1 - p_2))G_{\text{bulk}}(p_1, \omega_1).$$

Now we concentrate on the momentum non-preserving part. A diagram which remains connected when any of its internal line is cut is called one-particle irreducible. Any diagram with two external legs can be built by attaching one-particle irreducible diagrams after each other. For the momentum non-preserving diagrams we can separate the consecutive momentum preserving one-particle irreducible subdiagrams adjacent to a given external line. This gives subgraphs that are identical to the graphs in the series of the bulk propagator, so they give factors $G_{\text{bulk}}(p_1, \omega_1)$ and $G_{\text{bulk}}(p_2, \omega_2)$ at the external lines, respectively. The contributions of the remaining momentum non-preserving subgraphs are collected in $B(p_1, p_2, \omega_1)$. As a consequence, we have the following form for the propagator:

$$\begin{aligned} G(p_1, p_2, \omega_1) = {} & 2\pi(\delta(p_1 + p_2) + \delta(p_1 - p_2))G_{\text{bulk}}(p_1, \omega_1) \\ & + G_{\text{bulk}}(p_1, \omega_1)B(p_1, p_2, \omega_1)G_{\text{bulk}}(p_2, \omega_1). \end{aligned}$$

If in a Feynman graph, contributing to $B(p_1, p_2, \omega_1)$, both external lines are incident on a boundary vertex, then it does not depend on any of the momenta. Its contribution is collected in $B_3(\omega)$. Terms depending on one momentum only are collected in $B_2(p_1, \omega_1)$, these are the graphs in which one of the external lines ends in a boundary vertex. The contribution of a diagram starting with bulk vertices on both ends depends both on p_1 and p_2. Such contributions are collected in $B_1(p_1, p_2, \omega_1)$. Summing up all these terms we have (13).

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